EXACT SOLUTIONS OF CERTAIN CONVECTION PROBLEMS

PMM Vol. 32, No. 3, 1968, pp. 482-487

V.L. KATKOV
(Novosibirsk)

(Received January 4, 1968)

Invariant group solutions of certain problems defining convective motions are indicated. A part of solutions of this type had been previously obtained by other methods.

1. Plane motions of an incompressible viscous fluid resulting from a nonuniform heating of boundaries are considered. In a Cartesian system of coordinates such motions are defined by the system of Eqs [1]:

$$\begin{aligned} &uu_x + vu_y = u_{xx} + u_{yy} - p_x \\ &uv_x + vv_y = v_{xx} + v_{\;y} - p_y + \lambda\theta \\ &u\theta_x + v\theta_y = \sigma(\theta_{xx} + \theta_{yy}), \qquad u_x + v_y = 0 \end{aligned} \tag{1.1}$$

Here u and v are the velocity components along the x-, and y-axis, p is the pressure, θ the temperature (or more accurately, the divergence of these from certain standard values), λ is the Grasshoff number, and σ the Prandtl number.

The system of Eqs. (1.1) allows for a group of transformations to be carried out for the determination of which it is sufficient to know the appropriate basis operators [2]

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial p}, \quad X_4 = \lambda y \frac{\partial}{\partial p} + \frac{\partial}{\partial \theta} \\ &X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - 2p\frac{\partial}{\partial p} - 3\theta\frac{\partial}{\partial \theta} \end{aligned} \tag{1.2}$$

The first operator defines the displacement transformation in x:

$$x' = x + a, \quad y' = y, \quad u' = u, \quad v' = v, \quad p' = p, \quad \theta' = \theta$$

Here primes denote the new variables, and a is an arbitrary parameter. Two other operators similarily define displacement transformations in y and p. The fourth operator is equivalent to transformation

$$x' = x, \quad y' = y, \quad u' = u, \quad v' = v, \quad p' = p + \lambda y a, \quad \theta' = \theta + a$$

Finally, the fifth operator defines the stretching transformation

$$x' = ax, \; y' = ay, \; u' = \frac{u}{a}, \; v' = \frac{v}{a}, \; p' = \frac{p}{a^2}, \; \theta' = \frac{\theta}{a^3}$$

If the invariant group solution is subjected to a transformations from the basic group, a solution of system (1.1) is again obtained. It is therefore necessary to separate the essentially different solutions, i.e. those which cannot be converted one into another by any of the basic group transformations [2]. For the construction of essentially different solutions it is necessary to know the optimal system of single-parameter subgroups. Simple calculations show that such a system is generated by five operators

$$X_1, \quad \alpha X_1 + X_3, \quad \alpha X_1 + X_4, \quad \alpha X_1 + X_2 + \beta X_4, \quad X_5 \tag{1.3}$$

We shall denote by H_j the subgroup generated by operator X_j.

2. We shall consider the resulting solutions. The technique of deriving these was described in [2] and consists of the following. The invariants of subgroup H_j are determined as a set of linearly independent solutions of Eq. $X_j J = 0$. Let there be k, $J_1, J_2, \ldots, J_k$ of such invariants. A part of the invariants is then selected as independent variables, the remaining being considered to be functions of the former and are substituted into Eqs. (1.1). Integration of equations thus derived yields the required solution. Constants of integration arising in the process of integration will be denoted by u_i, v_i, θ_i ($i = 0.1$, p_0.

Due to lack of space we shall write down expressions for u, v and θ only.

2.1. Subgroup H_1. The invariants are $J_1 = u$, $J_2 = v$, $J_3 = p$, $J_4 = \theta$, $J_5 = y$; We look for a solution in the form $u = U(y)$, $v = V(y)$, $\theta = T(y)$, $p = P(y)$. After substitution into (1.1) we obtain from the last equation $v = v_0 =$ const, and the remaining take the form:

$$v_0 U' = U'', \qquad P' = \lambda T, \qquad v_0 T' = \sigma T'' \tag{2.1}$$

We shall first consider the case of $v_0 = 0$. After integration we obtain (compare with [3])

$$u = u_0' y + u_1, \quad v = 0, \quad \theta = \theta_0 y + \theta_1 \tag{2.2}$$

Solution (2.2) defines a Coutte flow between two moving planes, when the temperature of these are different [4].

If $v_0 \neq$ then

$$u = u_0 e^{v_0 y} + u_1, \quad v = v_0, \quad \theta = \theta_0 e^{my} + \theta_1 \tag{2.3}$$

where $m = v_0/\sigma$, will be the solution of (2.1). This solution defines the convection through two porous walls with a gas being fed through one wall and sucked away through the other, and the velocity vector at each of the walls having different angles of inclination [5].

2.2. Subgroup H_2. We look for a solution in the form $u = U(y)$, $v = V(y)$, $\theta = T(y)$. $p = \alpha x + P(y)$. As in the previous case it again follows from the last equation that $v = v_0$. The remaining equations coincide with (2.1), however the term $-\alpha$ appears in the right-hand side of the first of these. The solution for $v_0 = 0$ is

$$u = {}^1/_2\, \alpha y^2 + u_0 y + u_1, \qquad v = 0, \quad \theta = \theta_0 y + \theta_1 \tag{2.4}$$

This solution is a generalization of the Poiseuille flow [4] generated between two heated planes by the action of a constant pressure gradient.

Let now $v_0 \neq 0$. Integration of equations yields

$$u = u_0 e^{v_0 y} - \frac{\alpha}{v_0}\left(y + \frac{1}{v_0}\right) + u_1, \; v = v_0, \; \theta = \theta_0 e^{mv} + \theta_1, \quad m = \frac{v_0}{\sigma} \tag{2.5}$$

This solution may be interpreted as a flow between porous walls.

2.3. Subgroup H_3. We look for a solution in the form:

$$u = U(y), \; v = V(y), \; \theta = \beta x + T(y), \qquad p = \lambda\beta x y + P(y)$$

and after substitution into (1.1), we arrive at Eqs.

$$VU' = U'' - \lambda\beta y, \qquad VV' = V'' - P' + \lambda T, \qquad U\beta + VT' = \sigma T'', \qquad V' = 0 \tag{2.6}$$

The first case, with $v = 0$, yields solution

$$u = {}^1/_6 \lambda\beta y^3 + u_0 y + u_1, \quad \theta = \beta x + \frac{\beta}{\sigma}\left(\frac{\lambda\beta}{120} y^5 + \frac{u_0}{6} y^3 + \frac{u_1}{2} y^2\right) + \theta_0 y + \theta_1 \tag{2.7}$$

The derived solution defines, e.g., a flow inside of a flat channel along which is maintained a constant temperature gradient β (compare with [6]).

The solution for $v \neq 0$ is also written in elementary functions.

2.4. **Subgroup** H_4. Solutions which obtain in this subgroup depend on two parameters α and β which may vanish. There are therefore four discernible cases.

2.4.1. Let initially $\alpha = \beta = 0$. A solution is to be sought in the form:

$$u = U(x), \qquad v = V(x), \qquad \theta = T(x), \qquad p = P(x)$$

The system of Eqs. (1.1) takes the form

$$UU' = U'' - P', \qquad UV' = V'' + \lambda T, \qquad UT' = \sigma T'', \qquad U' = 0 \qquad (2.8)$$

It follows from the last equation that $u = u_0$. When $u_0 = 0$ integration of (2.8) yields

$$u = 0, \; v = -\lambda\,((^1/_6\theta_0 x^3 + {}^1/_2\theta_1 x^2) + v_0 x + v_1, \quad \theta = \theta_0 x + \theta_1 \qquad (2.9)$$

This solution defines, e.g., a flow in a vertical slot with different wall temperatures [7]. If however $u_0 \neq 0$, then

$$u = u_0, \qquad \theta = \theta_0 e^{mx} + \theta_1 \qquad (m = u_0 / \sigma)$$

$$v = v_1 e^{u_0 x} - \frac{\lambda\theta_0\sigma^2}{u_0^2(1-\sigma)}\, e^{mx} + \frac{\lambda\theta_1}{u_0}\, x + \frac{\lambda\theta_1}{u_0^2} + v_0 \qquad (2.10)$$

In order to interpret this solution we shall consider the following problem. Let there be at $x = 0$ a porous wall at temperature θ_0 through which air is drawn off at the rate u_0, and let in the right-hand half-plane at an infinitely great distance from this wall the air temperature be zero. The boundary conditions will consequently be

$$\begin{array}{lll} u = u_0, \; v = 0, \; \theta = \theta_0 & \text{for} \;\; x = 0 & \\ u = u_0, \;\; v = 0, \; \theta = 0 & \text{for} \;\; x \to \infty & \end{array} \quad (u_0 < 0)$$

Pressure is throughout constant and equal p_0. With these boundary conditions we obtain from (2.10)

$$v = \frac{\lambda\theta_0\sigma^2}{u_0^2(1-\sigma)}\,(e^{u_0 x} - e^{mx}), \;\; \theta = \theta_0 e^{mx} \qquad (2.11)$$

The streamlines are shown on Fig. 1.

2.4.2. The case of $\beta = 0$, $\alpha \neq 0$ was investigated in [8].

2.4.3. $\beta \neq 0$, $\alpha = 0$. We look for a solution in the form:

$$u = U(x), \qquad v = V(x), \qquad \theta = \beta y + T(x), \qquad p = {}^1/_2\lambda\beta y^2 + P(x)$$

After substitution into (1.1) we obtain system

$$UU' = U'' - P', \qquad UV' = V'' + \lambda T, \qquad UT' + \beta V = \sigma T'', \qquad U' = 0. \qquad (2.12)$$

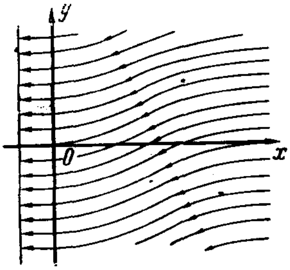

Fig. 1

From the last equation we have $u = u_0$, and from the first $P = p_0$. System (2.12) is equivalent to a single fourth order equation for V

$$V^{\mathrm{IV}} - u_0\left(1 + \frac{1}{\sigma}\right)V''' + \frac{u_0^2}{\sigma}V'' + \frac{\lambda\beta}{\sigma}V = 0 \qquad (2.13)$$

If V has been determined, then T is found from the second of Eqs. (2.12)

We shall first consider the case of $u_0 = 0$, $\beta > 0$. The characteristic equation corresponding to (2.13) is of the form

$$k^4 = -\omega^2 \qquad (\omega^2 = \lambda\beta / \sigma) \qquad (2.14)$$

The roots of Eq. (2.14) are

$$k_{1,2} = \mu(1 \pm i)$$
$$k_{3,4} = -\mu\,(1 \pm i), \qquad (\mu = {}^1/_2\,\sqrt{2\,|\,\omega\,|})$$

In order to interpret this solution we shall consider the following problem. Let there be at $x = 0$ a vertical wall the temperature of which linearly increases with increasing y, and let the gas be stationary at an infinitely great distance from this wall, i.e.

$$\begin{matrix} u = v = 0, & \theta = \beta y + \theta_0 & \text{for} & x = 0 \\ u = v = 0, & \theta = \beta y & \text{for} & x \to \infty \end{matrix} \qquad (\beta > 0)$$

Pressure is throughout constant. A solution which satisfies these conditions is

$$u = 0, \quad v = \frac{\lambda\theta_0}{2\mu^2}\, e^{-\mu x} \sin \mu x, \ \theta = \beta y + \theta_0\, e^{-\mu x} \cos \mu x \tag{2.15}$$

The whole flow (Fig. 2) has been split into a series of strips $L = \pi/\mu$ wide with the direction of flow reversed when passing from one strip to another. The total gas flow through the cross section is

$$Q = \int_0^\infty v dx = \frac{\lambda\theta_0}{4\mu^3}$$

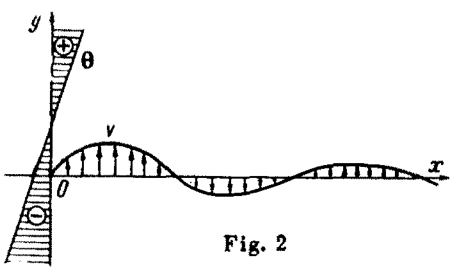

Fig. 2

We turn now to the case of $u = 0$, $\beta < 0$. The vertical temperature gradient is now negative. We denote $\lambda\beta/\sigma$ by $-\omega^2$; V may be written in the form:

$$V = C_1 e^{\mu x} + C_2 e^{-\mu x} + C_3 \sin \mu x + C_4 \cos \mu x \qquad (\mu = \sqrt{|\,\omega\,|})$$

Two problems may be considered here (compare with [9]). The first corresponds to conditions

$$u = v = 0, \ \ \theta = \beta y \quad \text{for} \quad x = 0, \qquad u = v = 0, \ \ \theta = \beta y \quad \text{for} \quad x = L$$

Parameter L, the slot width, is unknown. The solution is of the form

$$u = 0, \quad v = v_0 \sin \mu x, \quad \theta = \beta y + \frac{\mu^2}{\lambda}\, v_0 \sin \mu x \tag{2.16}$$

The slot width $L = k\pi/\mu$, $k = 1, 2, \ldots$ Magnitude v_0 defines the maximum value of velocity.

The second problem may define a flow in the vicinity of a 'contact discontinuity'. The boundary conditions are as follows:

$$u = 0, \ v = v_0, \ \theta = -\frac{\mu^2 v_0}{\lambda} + \beta y \quad \text{for} \ x = 0$$
$$u = 0, \quad v = 0, \quad \theta = \beta y \qquad \text{for} \ x \to \infty$$

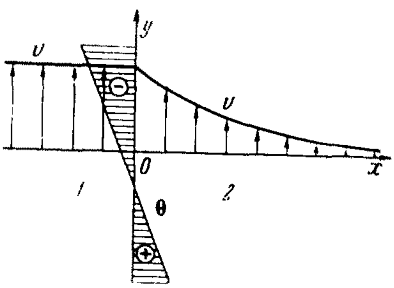

Fig. 3

With these boundary conditions the solution is of the form (Fig. 3)

$$u = 0, \ v = v_0\, e^{-\mu x}, \ \theta = \beta y - \frac{\mu^2 v_0}{\lambda}\, e^{-\mu x} \tag{2.17}$$

When $u_0 \neq 0$ then the solution may be written in the explicit form, but the formulas would be rather cumbersome.

2.4.4. The case of $\alpha \neq 0$, $\beta \neq 0$. We look for a solution in the form:

$$u = U(\xi), \quad v = V(\xi), \quad \theta = \beta y + T(\xi)$$
$$p = {}^1/_2\, \lambda\beta y^2 + P(\xi), \qquad (\xi = x - \alpha y)$$

The substitution of these expressions into (1.1) yields from the continuity equation the first integral

$$U = \alpha V + u_0 \tag{2.18}$$

Taking this integral into account we rewrite the remaining equations as follows

$$\alpha u_0 V' = -P' + \alpha(1+\alpha^2)V'', \qquad u_0 V' = \alpha P' + (1+\alpha^2)V'' + \lambda T$$
$$u_0 T' + \beta V = \sigma(1+\alpha^2)\, T'' \tag{2.19}$$

If constant $u_0 = 0$, then system (2.19) is reduced to the following simple equation for the temperature

$$T^{\mathrm{IV}} + \frac{\lambda\beta}{\sigma(1+\alpha^2)^3}\, T = 0 \tag{2.20}$$

Velocity is determine from the last equation of system (2.19). Solution (2.20) for $\beta < 0$ was analyzed by Prandtl [10] in connection with his investigation of convection above an inclined plane.

2.5. Subgroup H_5 yields the only one self-similar solution. It is convenient to change to polar coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad u = U\cos\varphi - V\sin\varphi, \quad v = U\sin\varphi + V\cos\varphi$$

Here U and V are the velocity vector radial and tangential components. A solution is sought in the form

$$U = \frac{1}{r}F(\varphi), \; V = \frac{1}{r}\Phi(\varphi), \; p = \frac{1}{r^2}P(\varphi), \; \theta = \frac{1}{r^3}T(\varphi) \tag{2.21}$$

From the continuity equation expressed in polar coordinates follows that $\Phi = u_0$. If $u_0 \neq 0$, then there will exist a stream of gas passing through the half-line ϕ = const. To simplify computations we shall assume in the following that $u_0 = 0$. It is readily demonstrable that in this case the streamlines will correspond to half-lines ϕ = const.

Eqs. (1.1) are expressed in polar coordinates, and solution (2.21) is substituted into these. After some simple computations we obtain system

$$F'' + F^2 + 2P + \lambda T\sin\varphi = 0, \quad T'' + 9T = -3\sigma^{-1}FT, \quad P' = 2F' + \lambda T\cos\varphi \tag{2.22}$$

We shall consider the following problem. A point heat source (or sink) with output Q is specified at the vertex of a plane angle the side walls of which are at different temperatures. The flow inside of this angle is to be determined. Therefore system (2.22) is to be complemented by the following boundary conditions

$$F = 0, \quad T = \theta_1 \qquad \text{for} \quad \varphi = \alpha_1 \tag{2.23}$$

$$F = 0, \; T = \theta_2, \; \int_{\alpha_1}^{\alpha_2} rU\,d\varphi = \int_{\alpha_1}^{\alpha_2} F\,d\varphi = Q, \; \text{for} \quad \varphi = \alpha_2$$

Here Q is the flow rate of gas. The problem thus stated may be interpreted in several ways.

With $\alpha_1 = 0$, $\alpha_2 = \pi$ and $Q > 0$ we have the problem of a burning gas jet having and infinitely high temperature at its source. When $\alpha_1 = -\frac{1}{2}\pi$, $\alpha_2 = 3/2\pi$ and $Q > 0$, we obtain an infinitely narrow slot (needle) instead of a plane angle from which flows a gas at either positive, or negative temperature.

We shall note the following fact. For $\lambda = 0$ Eqs. (2.22) with boundary conditions (2.23)

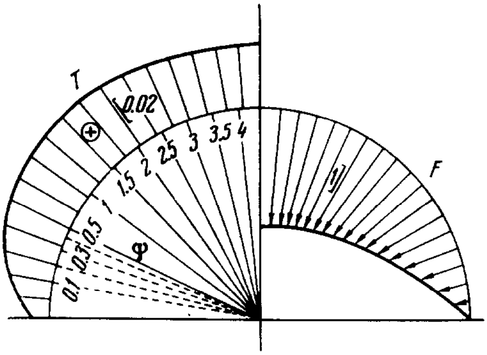

Fig. 4

define the Hamel solution for a diffuser [4], when its walls are at different temperatures. As is known, a source flow exists in this case only when cone angles are small, while in the case of a sink a solution does always exist. It seems that a similar conclusion applies to problem (2.22), (2.23), however it had not been possible to obtain a proof of this statement.

System (2.22) with boundary conditions (2.23) was solved numerically for the following parameter values:

$$\alpha_1 = 0, \quad \alpha_2 = \pi, \quad \theta_1 = \theta_2 = 0{,}01, \quad \lambda = \sigma = 1, \quad Q = -8.88.$$

Computations were carried out in accordance with the method described in [11], the results are shown on Fig. 4. At the right-hand side of this diagram is shown the angular distribution of radial velocity F; distribution of temperature T is shown in the upper part of the left-hand side, and the streamlines appear in the lower part of the left-hand side of the diagram. Figures indicate the rate of gas flow through the cross-section. In order to calculate the heat flux through the wall it is necessary to have the value of $T'(0)$ which in the adduced computations was found to be $|T'(0)| = 0.113$.

As previously noted, (2.22) defines the Hamel solution when $\lambda = 0$. We note that in this case the temperature may be specified by an arbitrary power law $\theta = r^m T(\varphi)$.

Eqs. (1.1) are reduced to a system as follows:

$$F'' + F^2 + 2P = 0, \qquad T'' + m^2 T = m\sigma^{-1} F T, \qquad P = 2F + p_0 \qquad (2.24)$$

where p_0 is an arbitrary constant. For $m = 0$ the temperature will be a linear function of ϕ, while F is implicitly expressed in terms of ϕ by means of a quadrature.

The author is grateful to G.Z. Gershuni and E.M. Zhukhovitskii for discussing this paper and for providing information as to the bibliography on known precise solutions.

BIBLIOGRAPHY

1. Landau, L.D., and Lifshits, E.M., Mechanics of Continua. 2nd ed. M., Gostekhizdat, 1954.
2. Ovsiannikov, L.V., Group Properties of Differential Equations. Novosibirsk, Izd. SO, Akad. Nauk, SSSR, 1962.
3. Schlichting, H., Einige exakte lösungen für die Temperaturverteilung in einer laminaren Strömung. ZAMM, Bd. 31, Hft. 3, 1951.
4. Slezkin, N.A., Dynamics of a Viscous Incompressible Fluid. M., Gostekhizdat, 1955.
5. Lee, J.M., and Gill, W.N., Heat transfer in laminar and turbulent flows between parallel plates with transverse flows. A.J. Ch. E. Journal, Vol. 10, No. 6, 1964.
6. Birikh, R.V., On the thermocapillar convection in a horizontal liquid layer. PMTF, No. 3, 1966.
7. Gershuni, G.Z., On the stability of a plane convective motion of a fluid. Zh.T.F., Vol. 23, No. 10, 1953.
8. Katkov, V.L., An exact solution in the theory of oceanic flows. Izv. Akad. Nauk SSSR, Physics of the atmosphere and ocean, Vol. 4, No. 1, 1968.

9. Yih, Chia-Shun., Thermal instability of viscous fluids. Quart. Appl. Math., Vol. 17, No. 1, 1959.

10. Prandtl, L., Hydro-Aerodynamics. M., (Russian translation). Izd. inostr. lit., 1949.

11. Katkov, V.L., Self-similar solution of the problem of local wind. Izv. Akad. Nauk USSR, Physics of the atmosphere and ocean. Vol. 1, No. 2, 1965.

Translated by J.J.D.